Optimum Fiber Angle of Unidirectional Composites for Load with Variations

Mitsunori Miki,* Yoshisada Murotsu,† and Tetsuo Tanaka‡
University of Osaka Prefecture, Sakai, Osaka 591, Japan

Optimum fiber orientations of undirectional composite materials under probabilistic loading conditions are found to be different from those under deterministic loading conditions. A simple and intuitive method called the interior tangent ellipsoid (ITE) method is proposed for optimum design of composites under the action of loads with variations. This method uses an analytical approach and enables a clear interpretation of the optimum orientation angles for probabilistic conditions. A comparison between the ITE method and the advanced first-order second-moment (AFOSM) method is made, and good agreements on optimum fiber orientations are observed. The computation time for the ITE method is much less than the time for the AFOSM method. It is concluded that the proposed method is valid for determining the optimum fiber angles of undirectional composites under probabilistic loading conditions and the resulting optimum designs are much different from those under deterministic conditions.

Nomenclature

CV = coefficient of variation
$CV(X)$ = coefficient of variation of X
d = size of ellipse
$d_{\min}$ = minimum value of d
$E(X)$ = mean value of X
F_{ij}, F_i = coefficients in failure criterion
F^*_{xy} = normalized interaction coefficient in failure criterion
R = strength ratio
R_s = longitudinal shear strength
R_x, R_y = tensile strength
R'_x, R'_y = compressive strength
SD = standard deviation
S_i, Si = stress
S_{ia} = allowable stress
u_i = transformed variable of S_i in the standard normal space
v_i = transformed variable of u_i by translation
w_i = transformed variable of v_i by rotation
x = axis along fibers
y = axis perpendicular to fiber direction
γ = correlation coefficient between S_1 and S_2
θ = fiber orientation angle
μ_i = mean value of applied stress S_i
σ_i = standard deviation of applied stress S_i

Subscripts

s = shear in x-y axes
x = longitudinal
y = transversal
1 = major reference axis
2 = minor reference axis, perpendicular to 1 axis
6 = shear in 1-2 axes

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*Associate Professor, Department of Aeronautical Engineering. Member AIAA.
†Professor, Department of Aeronautical Engineering. Member AIAA.
‡Graduate Student.

Introduction

THE determination of fiber orientation angles is the most important subject in optimum design of composite materials. Many investigations have been carried out on it,[1-4] but most of them yield optimum fiber angles for deterministic conditions where strengths and loads have no variations. However, those results are considered to be invalid for probabilistic conditions where strengths and loads have some variations.

Cederbaum et al.[5] proposed a method to evaluate the reliability of each ply in laminated plates subjected to in-plane random static loads, and Miki et al.[6] developed the general method for optimum design of unidirectional fibrous composites under probabilistic conditions and found that the optimum fiber angle that yields the maximum reliability changes with the increase in the variation of applied load. This suggests that the optimum fiber angles under probabilistic conditions are different from those under deterministic conditions. The objectives of this study are 1) to find the optimum fiber orientation angles of undirectional composites for probabilistic conditions, and 2) to propose a simple method for the determination of the optimum angle. The motivation for the second is that the calculation based on the advanced first-order second-moment (AFOSM) method[7] is rather sophisticated and it requires much time.

This paper proposes a simple and intuitive method called the interior tangent ellipsoid (ITE) method, which enables an approximate reliability analysis using a deterministic and analytical approach. Presented are the concept of the ITE method, the analysis of the optimization problem, the numerical results, the comparison between the results calculated by the ITE and AFOSM methods, and the effectiveness and limits of the present method. It is concluded that the proposed method is valid for determining the optimum fiber angles of unidirectional composites under probabilistic loading conditions and the resulting optimum designs are much different from those under deterministic conditions. It is suggested that the variations of strengths and applied loads should be taken into account in the optimum design of composites.

Optimum Fiber Orientation Under Deterministic Load

Failure Criterion for Composites

There are some criteria proposed for the failure of unidirectional fibrous composites with respect to their principal axes. Among those, the Tsai-Wu criterion[8] has been used by many researchers, and it is recognized as the most general criterion for unidirectional composites. The Tsai-Wu criterion has the form

$$F_{xx}S_x^2 + 2F_{xy}S_xS_y + F_{yy}S_y^2 + F_{ss}S_s^2 + F_xS_x + F_yS_y = 1 \tag{1a}$$

where

$$F_{xx} = 1/R_xR'_x, \qquad F_x = 1/R_x - 1/R'_x$$
$$F_{yy} = 1/R_yR'_y, \qquad F_y = 1/R_y - 1/R'_y$$
$$F_{ss} = 1/R_s^2, \qquad F_{xy} = F^*_{xy}\sqrt{F_{xx}F_{yy}} \tag{1b}$$

The factor F^*_{xy} is assumed to be $-\frac{1}{2}$. The Tsai-Wu criterion is used in this paper, but any other criterion can be used in the proposed method.

For failure under any plane stress condition, off-axis (not along the material principal axes) stresses are transformed into on-axis (along the principal axes) stresses to use on-axis failure criteria. The coordinate systems are shown in Fig. 1, where 1 and 2 represent reference axes and θ is the angle between 1 and x axes. The stress transformation is expressed as

$$\begin{bmatrix} S_x \\ S_y \\ S_s \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} \tag{2a}$$

where

$$m = \cos\theta, \qquad n = \sin\theta \tag{2b}$$

A failure envelope on the S_1–S_2 plane with shear stress S_6 being zero is schematically shown in Fig. 2. The failure envelope becomes an ellipse when the Tsai-Wu criterion is used. Failure of a ply occurs when its ply stresses are represented by a point outside the envelope, and a ply is intact when its stress state is inside it.

Strength Ratio

Tsai proposed a method for determining the optimum fiber angle in terms of a strength ratio.[1] The strength ratio R is

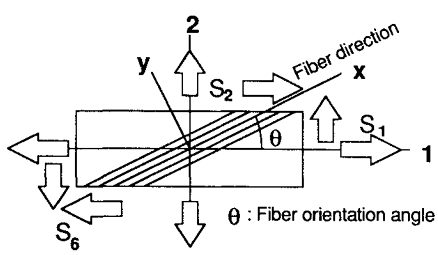

Fig. 1 Coordinate systems for unidirectional composites.

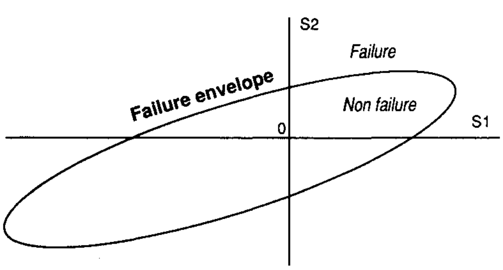

Fig. 2 Failure envelope.

defined by

$$S_{ia} = RS_i \qquad (i = 1,2,6) \tag{3}$$

The design method based on the strength ratio assumes that the direction of the applied load is constant. Figure 3 illustrates a failure envelope and the applied stress represented by point P. The optimum fiber angle is determined such that the strength ratio representing the ratio of the distance OQ to OP is maximized.

The optimum fiber orientation angle of unidirectional composite plates subject to an in-plane stress can be calculated analytically and the result is shown in Fig. 4 where the applied shear stress is zero. It should be noted that the optimum fiber angle is 0 deg for $S_1 > S_2$ and 90 deg for $S_1 < S_2$. For $S_1 = S_2$, any angle is optimum.

This simple result is easily explained in Fig. 5 where failure envelopes for 0 and 90 deg and angle θ (arbitrary) are drawn. When $S_1 > S_2$, and a stress state is represented by point P_1, the strength is maximized by taking the failure point at point R_0, which is on the 0-deg failure envelope.

The design method based on the strength ratio is valid as long as the applied loading is proportional and the failure envelope is deterministic. It does not, however, give the optimum solution for probabilistic conditions where the applied stresses are subject to uncertainty and the strengths have some variations. The discussion is materialized for the failure en-

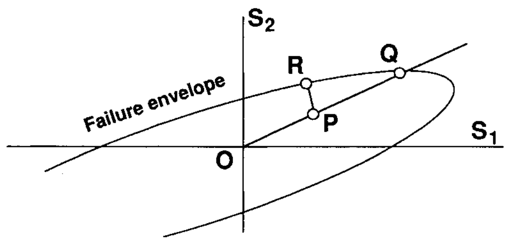

Fig. 3 Loading path and failure point.

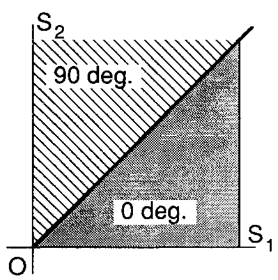

Fig. 4 Optimum fiber angles on the first quadrant of the applied stress plane, determined by the strength ratio method.

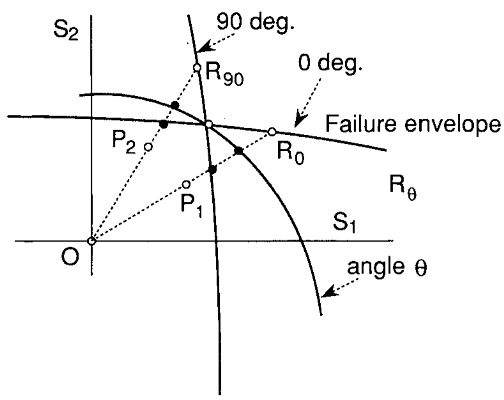

Fig. 5 Explanation of the result shown in Fig. 4.

velope given in Fig. 3. Point Q is the failure point for proportional loading and point R is the most dangerous point in the sense that the allowance for the variation of the loads is minimum. It is obviously recognized that the optimum fiber angle should be determined by a method based on the failure point R instead of point Q when the loading conditions are probabilistic.

Reliability of Composite Materials

The optimum fiber orientation angle of composite plates for probabilistic conditions can be obtained by using the AFOSM method[7] where failure probability P_f is evaluated by using the safety index β and the standard normal distribution function Φ, i.e., $P_f = \Phi(-\beta)$. The authors have applied the method for evaluating the reliability of unidirectional composites and found that the fiber angle yielding the maximum reliability changed with an increase in the variation of applied loads.[6]

One of our results is reproduced in Fig. 6, where the safety index of a unidirectional ply is calculated as a function of the orientation angle under an in-plane one-axis stress (S_1) and shear stress (S_6) condition. The mean value of S_1 is taken to be 0.3 GPa with its coefficient of variation being zero, whereas the variance of S_6 is changed keeping its mean value of 0.25 GPa. The figure also illustrates the effect of the variation in the applied shear stress S_6 on the reliability of the composite. Note that the material constants of the composite used for calculations are given in Table 1.

The angle that gives a maximum reliability is about 30 deg when the coefficient of variation CV of S_6 is zero. However, the angle increases with increase in CV. That is, optimum orientation angles differ not only with the mean values but also with the variations of the applied loads.

It is clearly understood that the consideration of the variations in the strength of composites and the applied load plays a very important role in optimum material design. However, it is not easy to calculate the probability of failure or the safety index of composites.

Typical methods for evaluating reliability are 1) a numerical integration, 2) a Monte Carlo simulation, 3) the first-order second-moment (FOSM) method,[9] and 4) the AFOSM method. The first and second methods are not always practical due to computational efficiency. The third method yields analytical results for many cases, but the accuracy of the solution obtained by the method decreases remarkably as the probability of failure decreases.[6]

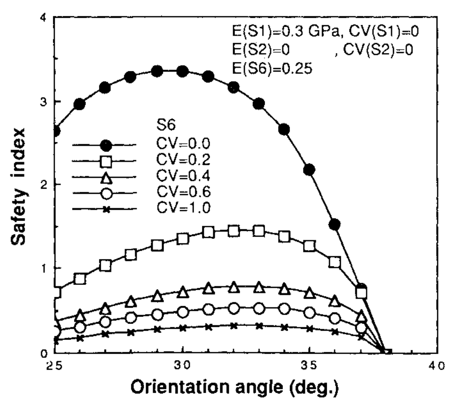

Fig. 6 Effect of the coefficient of variation for the shear stress on the reliability of a unidirectional composite.

On the contrary, the AFOSM method is useful and it is widely used in the structural reliability field. However, the method is rather too sophisticated for those whose major fields are not reliability, and it takes much computation time to provide safety index since the calculation is usually based on an optimization technique.

From those points of view, we propose a simple and intuitive method called the interior tangent ellipsoid method, which enables an approximate reliability analysis using a deterministic approach.

Interior Tangent Ellipsoid Method

Concept

When the strengths and the applied loads of unidirectional composites have some variations, a probabilistic failure envelope and the probability density of applied stresses are schematically shown in Fig. 7. The dark portion of the figure means high probability density. The reliability of composite materials is evaluated considering the overlap between these distributions. A typical evaluation method is the AFOSM method, as mentioned earlier.

The variations of applied loads are assumed to be considerably large in comparison with those of strengths of composites. Then, the variations of the strengths can be ignored. The failure is analyzed with the nominal failure envelope obtained from a failure criterion with deterministic material properties and the probability density of the applied stresses. Figure 8 illustrates the situation where the Tsai-Wu criterion is applied with the applied shear stress being zero. A point that has highest probability of failure is defined on the nominal failure envelope and, here, it is called the most dangerous point.

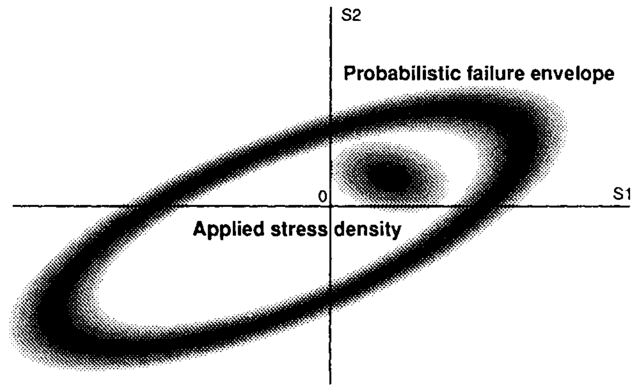

Fig. 7 Interaction between failure envelope and applied stresses under probabilistic conditions.

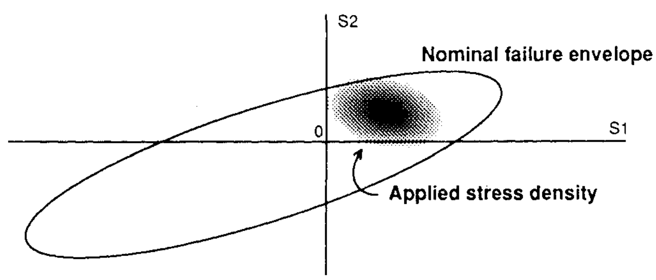

Fig. 8 Deterministic strengths and probabilistic applied stresses.

Table 1 Material constants used, MPa[8]

Material	Type	R_x	R'_x	R_y	R'_y	R_s
T300/5208	Graphite/epoxy	1500	1500	40	246	68

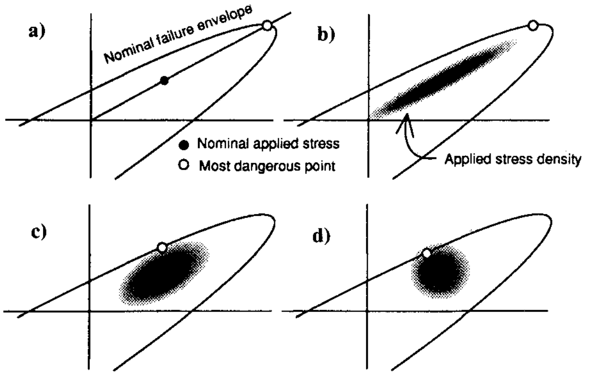

Fig. 9 **Most dangerous points for some cases where the probability distributions of applied stresses are different.**

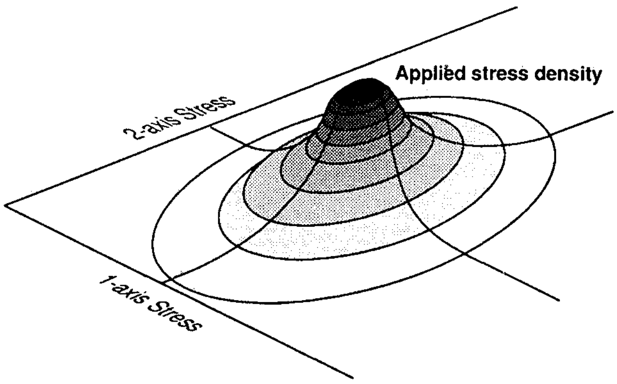

Fig. 10 **Joint probability density of applied stresses.**

The location of the most dangerous point changes according to the variations of the applied loads even if their mean values are constant. It makes the difference between deterministic and probabilistic conditions. Typical examples of the most dangerous points are shown in Fig. 9, where only the variations of the applied loads are different. A loading where stress components vary with the coefficients of correlation being unity and with the ratio of the coefficients of variation being constant corresponds to proportional loading. In the case of proportional loading, the most dangerous point is on the extension of the line connected from the origin to the point of the mean applied stresses. If the applied stresses are randomly distributed, the most dangerous point moves according to the shape of the distribution, as shown in Fig. 9.

It is easily seen that the most dangerous point is a tangent point between the nominal failure envelope and a contour curve of the probability density of the applied loads. A norm defined by the most dangerous point and the mean stress point can be used as a relative measure of reliability.

If the applied stresses follow the normal distribution, the contour curves of its density become ellipses, as shown in Fig. 10. The ellipticity and the inclination of the ellipse are related to the ratio of the standard deviations and the correlation coefficient of the applied stresses. Consequently, the problem of seeking the most dangerous point is to obtain an interior tangent ellipse of the nominal failure envelope, as shown in Fig. 11. The ellipse becomes an ellipsoid when the shear component of the applied stress is considered. It is found that a closed-form representation of the most dangerous point can be obtained as long as the nominal failure envelope is linear or quadratic. This method is called the interior tangent ellipsoid method. The method provides analytical evaluation of reliability, and an optimum orientation angle is obtained with minimum computation time.

When the applied stresses have the same values of the standard deviations with no correlation, the contour curves

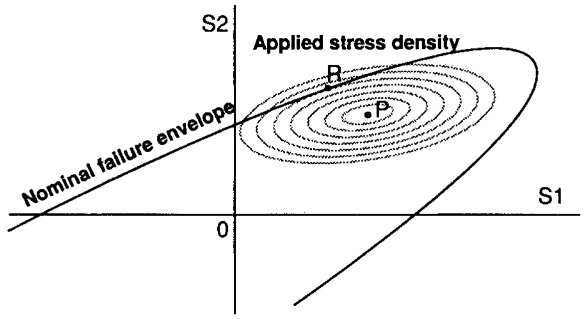

Fig. 11 **Concept of the ITE method: the most dangerous point is represented by point R.**

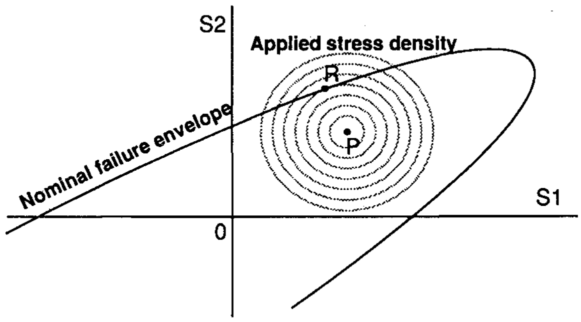

Fig. 12 **Joint probability density for the case where the standard deviations of the stresses are the same and there is no correlation between them.**

of the joint probability density become circles (or spheres). In this case, the optimum fiber angle is obtained by maximizing the distance between the mean stress point and the envelope, as shown in Fig. 12.

Analysis

The shear component of the applied load is assumed to be zero, for simplicity,

$$S_6 = 0 \tag{4}$$

Then, Eqs. (2) become

$$S_x = S_1 \cos^2\theta + S_2 \sin^2\theta \tag{5a}$$

$$S_y = S_1 \sin^2\theta + S_2 \cos^2\theta \tag{5b}$$

$$S_s = -S_1 \cos\theta \sin\theta + S_2 \cos\theta \sin\theta \tag{5c}$$

Substituting Eq. (5) into Eq. (1) yields

$$AS_1^2 + BS_2^2 + CS_1S_2 + DS_1 + ES_2 - 1 = 0 \tag{6a}$$

where

$$A = \frac{1}{R_xR_x'}\cos^4\theta + \frac{1}{R_yR_y'}\sin^4\theta + \left(\frac{1}{R_s^2} - \frac{1}{\sqrt{R_xR_x'R_yR_y'}}\right)\sin^2\theta\cos^2\theta \tag{6b}$$

$$B = \frac{1}{R_xR_x'}\sin^4\theta + \frac{1}{R_yR_y'}\cos^4\theta + \left(\frac{1}{R_s^2} - \frac{1}{\sqrt{R_xR_x'R_yR_y'}}\right)\sin^2\theta\cos^2\theta \tag{6c}$$

$$C = 2\left(\frac{1}{R_xR'_x} + \frac{1}{R_yR'_y} - \frac{1}{R_x^2}\right)\sin^2\theta\cos^2\theta$$
$$- \frac{1}{\sqrt{R_xR'_xR_yR'_y}}(\sin^4\theta + \cos^4\theta) \tag{6d}$$

$$D = \left(\frac{1}{R_x} - \frac{1}{R'_x}\right)\cos^2\theta + \left(\frac{1}{R_y} - \frac{1}{R'_y}\right)\sin^2\theta \tag{6e}$$

$$E = \left(\frac{1}{R_x} - \frac{1}{R'_x}\right)\sin^2\theta + \left(\frac{1}{R_y} - \frac{1}{R'_y}\right)\cos^2\theta \tag{6f}$$

The contour curve of the two-dimensional normal density is expressed as

$$\frac{(S_1 - \mu_1)^2}{\sigma_1^2} - \frac{2\gamma(S_1 - \mu_1)(S_2 - \mu_2)}{\sigma_1\sigma_2} + \frac{(S_2 - \mu_2)^2}{\sigma_2^2} = d \tag{7}$$

If the applied stresses have no correlation, that is, $\gamma = 0$, Eq. (7) becomes

$$\frac{(S_1 - \mu_1)^2}{\sigma_1^2} + \frac{(S_2 - \mu_2)^2}{\sigma_2^2} = d \tag{8}$$

Then, the problem is to obtain an ellipse, which is given by Eq. (8) and is interiorly tangent to the envelope represented by Eqs. (6). However, the problem cannot be analytically solved since the order of the derived equation becomes 6. Therefore, transformations are required to get an analytical solution.

At first, the following standardization is performed,

$$u_1 = \frac{S_1 - \mu_1}{\sigma_1}, \qquad u_2 = \frac{S_2 - \mu_2}{\sigma_2} \tag{9}$$

Substituting Eqs. (9) into Eqs. (6) and (8) yields

$$Fu_1^2 + Gu_2^2 + Hu_1u_2 + Iu_1 + Ju_2 + K = 0 \tag{10a}$$

where

$$F = A\sigma_1^2 \tag{10b}$$
$$G = B\sigma_2^2 \tag{10c}$$
$$H = C\sigma_1\sigma_2 \tag{10d}$$
$$I = 2A\sigma_1\mu_1 + C\sigma_1\mu_2 + D\sigma_1 \tag{10e}$$
$$J = 2B\sigma_2\mu_2 + C\sigma_2\mu_1 + E\sigma_2 \tag{10f}$$
$$K = A\mu_1^2 + B\mu_2^2 + C\mu_1\mu_2 + D\mu_1 + E\mu_2 - 1 \tag{10g}$$

and

$$u_1^2 + u_2^2 = d \tag{11}$$

A translation is performed so that the center of the ellipse given by Eqs. (10) is removed to the origin. The translation takes the form

$$v_1 = u_1 - u_{10}, \qquad v_2 = u_2 - u_{20} \tag{12a}$$

where

$$u_{10} = \frac{2GI - HJ}{H^2 - 4FG}, \qquad u_{20} = \frac{2FJ - HI}{H^2 - 4FG} \tag{12b}$$

where u_{10} and u_{20} are the coordinates of the center of the ellipse. Then, Eqs. (10) are rewritten as

$$Fv_1^2 + Gv_2^2 + Hv_1v_2 + L = 0 \tag{13a}$$

where

$$L = K - \frac{F(2GI - HJ)^2 + G(2FJ - HI)^2 + H(2GI - HJ)(2FJ - HI)}{(H^2 - 4FG)^2} \tag{13b}$$

Eq. (11) becomes

$$(v_1 + u_{10})^2 + (v_2 + u_{20})^2 = d \tag{14}$$

Next, the ellipse given by Eqs. (13) is rotated so that its cross-product term may vanish,

$$v_1 = w_1\cos\varphi - w_2\sin\varphi, \qquad v_2 = w_1\sin\varphi + w_2\cos\varphi \tag{15a}$$

where

$$\varphi = \pm\tfrac{1}{2}\sin^{-1}\left[\frac{H}{\sqrt{(F - G)^2 + H^2}}\right] \tag{15b}$$

Then, Eqs. (13) are expressed in the form

$$\frac{w_1^2}{a^2} + \frac{w_2^2}{b_2} = 1 \qquad (0 \le w_1 \le a,\ 0 \le w_2 \le b) \tag{16a}$$

where

$$\frac{1}{a^2} = -\frac{1}{L}(F\cos^2\varphi + G\sin^2\varphi + H\sin\varphi\cos\varphi) \tag{16b}$$

$$\frac{1}{b^2} = -\frac{1}{L}(F\sin^2\varphi + G\cos^2\varphi - H\sin\varphi\cos\varphi) \tag{16c}$$

Similarly, Eq. (14) is written as

$$(w_1 - w_{10})^2 + (w_2 - w_{20})^2 = d \qquad (0 \le w_{10}, 0 \le w_{20}) \tag{17a}$$

where

$$w_{10} = -u_{10}\cos\varphi - u_{20}\sin\varphi \tag{17b}$$

$$w_{20} = u_{10}\sin\varphi - u_{20}\cos\varphi \tag{17c}$$

Thus, the transformed nominal failure envelope and the applied stress ellipse are illustrated in Fig. 13. If the center of the applied stress ellipse is not in the first quadrant, the symmetric property of the figure is taken into account to apply the proposed method.

From Eqs. (16) and (17), the size of the ellipse d is expressed in terms of w_1:

$$d(w_1) = \left(1 - \frac{b^2}{a^2}\right)w_1^2 - 2w_{10}w_1 - 2bw_{20}\sqrt{1 - \frac{w_1^2}{a^2}} + w_{10}^2 + w_{20}^2 + b^2 \tag{18}$$

Observing the functional form of $d(w_1)$, it is concluded that the minimum value of d is given by

$$d_{\min} = \mathrm{Min}\,\{d(0), d(a), d(w_1^*)\} \tag{19}$$

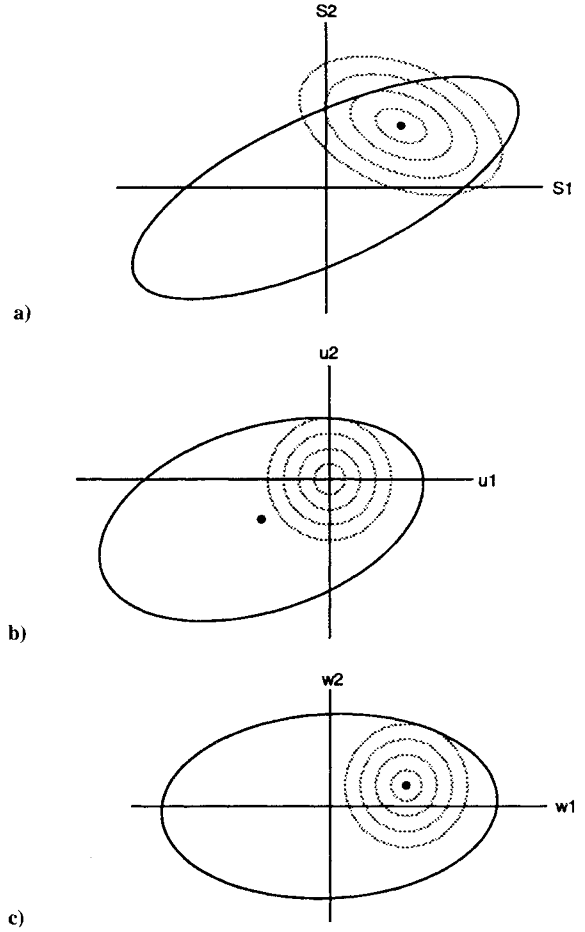

Fig. 13 Transformation processes for analytical treatment: a) initial state; b) standardization; c) translation and rotation.

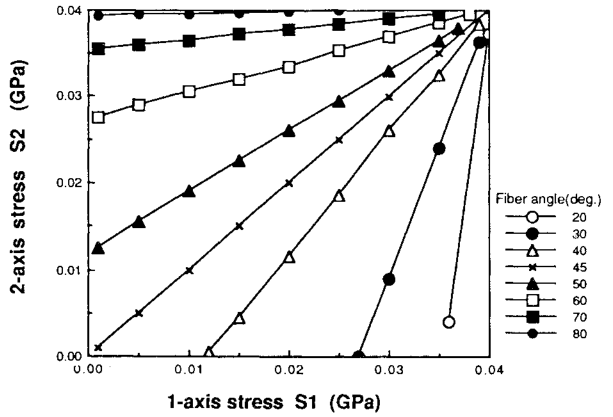

Fig. 14 Optimum fiber orientation angles of unidirectional composites obtained by the ITE method. The applied shear stress is set to zero. A contour line indicates a set of the mean stresses for which a certain fiber orientation angle is optimum.

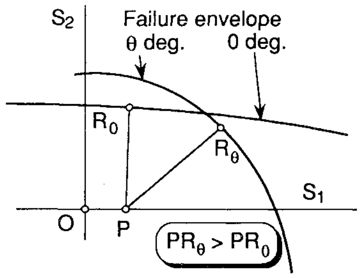

Fig. 15 Explanation for the result that the optimum fiber angle is not zero even for uniaxial loading condition: PR_θ = the distance from the mean stress point P to the failure envelope of θ deg; PR_0 = the distance from P to the failure envelope of 0 deg.

where $d(w_1^*)$ denotes an extreme value of d. Differentiating d with respect to w_1 and putting it to zero yield

$$Mw_1^4 + Nw_1^3 + Ow_1^2 + Pw_1 + Q = 0 \tag{20a}$$

where

$$M = (a^2 - b^2)^2 \tag{20b}$$

$$N = -2a^2w_{10}(a^2 - b^2) \tag{20c}$$

$$O = -a^2\{(a^2 - b^2)^2 - a^2w_{10}^2 - a^2b^2w_{20}^2\} \tag{20d}$$

$$P = 2a^4w_{10}(a^2 - b^2) \tag{20e}$$

$$Q = -a^6w_{10}^2 \tag{20f}$$

The final equation becomes quartic and its analytical solution can be obtained. The solution to Eqs. (20) gives an extreme point w_1^*.

Results and Discussion

Results by the Interior Tangent Ellipsoid Method

The material constants used for the calculation are listed in Table 1. The strengths are assumed to be deterministic and the applied stresses are assumed to be normally distributed.

Optimum fiber orientation angles of unidirectional composites obtained by the ITE method are illustrated in Fig. 14, where the abscissa and the ordinate represent the mean value of applied stresses. Only the ratio of the variations of the applied stresses plays an important role in the ITE method, whereas their magnitudes do not affect the optimum angle.

The following points are observed from the figure:

1) Optimum fiber orientations under probabilistic conditions are completely different from those under deterministic conditions.

2) Optimum orientations under probabilistic conditions depend on the mean stress level.

3) Optimum orientations under probabilistic conditions are not necessarily 0 or 90 deg, even if the mean values of the stresses are uniaxial.

4) Optimum orientations under probabilistic conditions tend to be identical to the deterministic results as the mean values of the applied stresses increase.

The second and third points just mentioned are easily explained by using Fig. 15 where the distance PR_θ is larger than the distance PR_0, that is, the most dangerous point is farthest for a certain orientation.

Comparison with the Advanced First-Order Second-Moment Method

The optimum fiber orientation angle obtained by using the AFOSM method is shown in Fig. 16, where the strengths have no variation and the values of the standard deviation of the applied stresses are 10 MPa each. It is found that the result by the ITE method is almost the same as those by the AFOSM method. Consequently, the ITE method is found to be valid for optimum design of composite materials for probabilistic loading conditions. It should be remarked here that only the AFOSM method provides the contours of equal β value, as shown in Fig. 16.

An advantage of the ITE method lies in the computation time for the evaluation of reliability. The ITE method does not provide an absolute value of reliability, but gives a relative

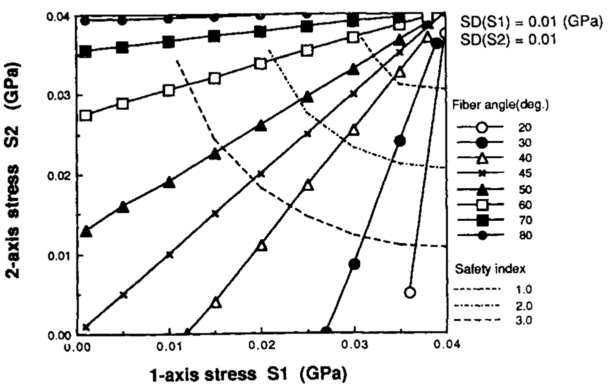

Fig. 16 Optimum fiber orientation angles of unidirectional composites obtained by the AFOSM method. The standard deviations of the applied stress are 10 MPa each. Dotted lines indicate contours of safety indices being constant.

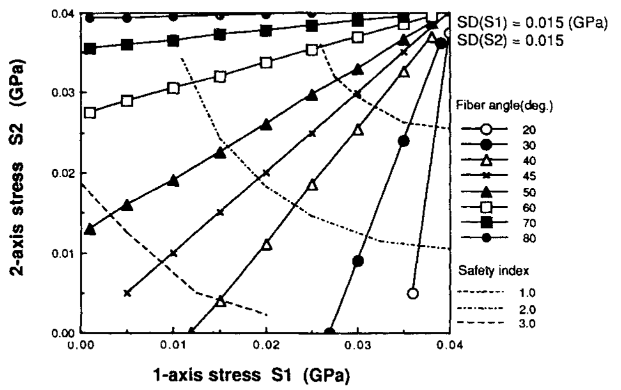

Fig. 17 Optimum angles obtained by the AFOSM method for the standard deviations of the applied stresses being 15 MPa each.

measure. The comparison of the computation times between the ITE and AFOSM methods are listed in Table 2. It is noted that the calculation time for the ITE method is one hundredth of the time for the AFOSM method. In the AFOSM method much time is required to search for a global minimum since the limit state function derived from the failure criterion is multimodal.

Effect of Standard Deviations of Applied Stresses

The ITE method takes account of only the ratio of the standard deviations of the applied stresses, so that the effect of their magnitudes on the optimum angle is discussed here. Figure 17 shows the results by the AFOSM method where the standard deviations of the applied stresses are 15 MPa each, whereas Fig. 16 gives the results for 10 MPa. It is found from these figures that the optimum orientations do not change with the magnitudes of the standard deviation. Therefore, the ITE method is confirmed to be effective from this point.

Effect of Ratio of Standard Deviations of Applied Stresses

The ratio of the standard deviations of applied stresses affects optimum orientations since it determines the ellipticity of the contour curve of the probability density of the stresses. The comparison between the results by the ITE and AFOSM methods for $\sigma_1:\sigma_2 = 2:1$ is given in Figs. 18. It is clearly observed that both results are very similar and the effectiveness of the ITE method is also confirmed.

Effect of Variations of Strengths

The ITE method uses nominal failure envelopes of composites; that is, the variations of the strengths are ignored. Therefore, the ITE method is effective only for the case where the variations of the applied stresses are much larger than those of the strengths. Otherwise, optimum orientations change, as shown in Figs. 19, where the coefficients of variation of the strengths are 5 and 10%, whereas Fig. 16 illustrates the result for 0%; that is, no variations exist in the strengths.

It is observed that the optimum orientations for probabilistic conditions approach those for deterministic conditions, shown in Fig. 4, as the variations in the strengths become large. This means that the variations of the stresses can be ignored when the variations in the strengths are relatively large, i.e., the deterministic optimum design is applicable to the determination of optimum orientation angles.

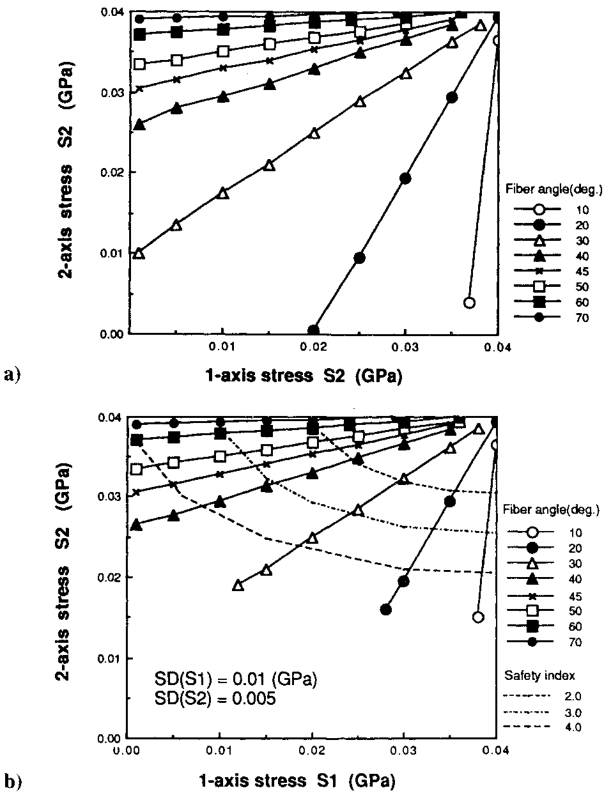

Fig. 18 Comparison between the optimum angles obtained by the ITE and AFOSM methods for the case where the ratio of the standard deviations of the applied stresses is 2 to 1: a) ITE method; b) AFOSM method.

Table 2 Comparison between the ITE and AFOSM methods

Mean value of applied stress, GPa		Standard deviation of applied stress, GPa		ITE		AFOSM	
*S*1	*S*2	*S*1	*S*2	Optimum angle, deg	Calculation time, s	Optimum angle, deg	Calculation time, s
0.03	0.03	0.005	0.005	45.0	0.633	45.0	69.5
0.04	0.02	0.005	0.005	9.0	0.667	9.0	67.5
0.03	0.02	0.007	0.005	27.0	0.617	27.0	69.1

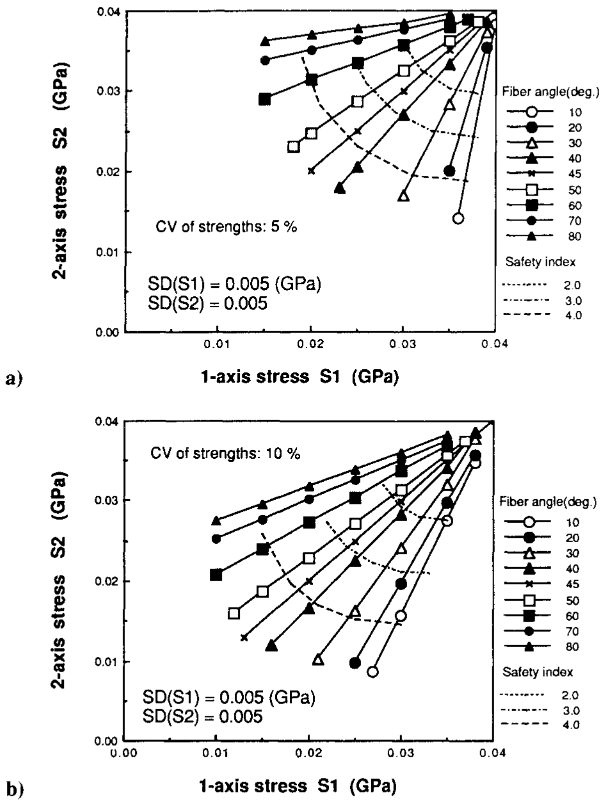

Fig. 19 Effects of the variation in the strengths on the optimum orientation angles evaluated by the AFOSM method: a) $CV = 5\%$; b) $CV = 10\%$.

Conclusions

A simple method is proposed for determining optimum fiber orientation angle of composites under probabilistic loading conditions. The method is called the ITE method and it is demonstrated to be useful, comparing the results by the proposed method and the AFOSM method previously developed by the present authors. The advantage of the ITE method lies in the analytical evaluation of reliability for unidirectional composites, which enables a clear interpretation of the optimum orientation angles under probabilistic conditions. The method can be extended to the optimum design of laminates. The relationship between the optimum orientations for deterministic and probabilistic conditions is also clarified through numerical examples.

Acknowledgments

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